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THEORY OF A PERIODIC LAMINAR BOUNDARY LAYER

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A method is proposed for the analysis of a periodic laminar boundary layer, refining the conventional methods of Lin, Rayleigh, and Hill and Stenning and providing a basis for the unification of those methods.

Derivation of the Fundamental System of Equations. The equations for a periodic laminar boundary layer have the form

$$\left.\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2},\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\end{aligned}\right\} \tag{1}$$

$$u=0;\ v=0 \text{ at } y=0;\ u\to U(x,\ t) \text{ as } y\to\infty;$$
$$u=f(y,\ t) \text{ at } x=x_f. \tag{2}$$

The velocity at the outer boundary of the boundary layer is given by the expression

$$U(x,\ t)=U_0(x)+W(x)\cos(\omega t).$$

The absence of a temporal boundary condition in the case of steady-state periodic motion renders it impossible, in principle, to solve the problem directly. This fact makes it necessary to adopt a specific representation of the time dependence of the functions u and v.

We investigate the expansions of these functions in Fourier series, written in complex form:

$$\left.\begin{aligned}u=u_0(x,\ y)+\mathrm{Re}\cdot\sum_{s=1}^{\infty}u_s(x,\ y)\exp(si\omega t),\\ v=v_0(x,\ y)+\mathrm{Re}\cdot\sum_{s=1}^{\infty}v_s(x,\ y)\exp(si\omega t).\end{aligned}\right\} \tag{3}$$

Here u_0 and v_0 are unknown real functions, and u_s and v_s are unknown complex functions. The functions u and v can be represented by Fourier series, since they satisfy the sufficient conditions for expansion (periodicity with respect to time and differentiability at any point of the domain of definition); see the system (1) and the boundary conditions (2). Assuming sufficiently rapid convergence of the series (3), hereinafter we use segments thereof containing only two harmonics. We substitute these segments into the system (1), writing the velocity at the outer boundary of the boundary layer in the form $U(x, t) = U_0(x) + \mathrm{Re}[W(x)\cdot\exp(i\omega t)]$. To take the operator Re for extraction of the real part outside the multiplication sign, we invoke the formula

$$\mathrm{Re}z_1\mathrm{Re}z_2=\frac{1}{2}\mathrm{Re}(z_1z_2+z_1\bar{z}_2).$$

The overbar is used everywhere to denote the complex conjugate, and z_1 and z_2 denote arbitrary complex numbers. After the appropriate calculations, the first equation of the system (1) can be written

$$\sum_{p=0}^{4}\mathrm{Re}[N_p(u_0,\ u_1,\ u_2,\ v_0,\ v_1,\ v_2)\exp(pi\omega t)]=0, \tag{4}$$

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where N_p denotes differential operators, the specific form of which will be given below. The latter sum consists of five terms in connection with the nonlinearity of the first equation of the system (1). From (4) we deduce

$$\sum_{p=0}^{4}[\operatorname{Re} N_p \cos(p\omega t) - \operatorname{Im} N_p \sin(p\omega t)] = 0.$$

Making use of the property of linear independence of the trigonometric functions, we obtain

$$\operatorname{Re} N_0 = 0, \quad N_p = 0 \; (p = 1, \ldots, 4). \tag{5}$$

The second equation of the system (1) is transformed analogously. We go over to dimensionless variables in the system (5) according to the formulas

$$x = Lx', \; y = \delta y', \; y = \delta_k \xi', \; U_0 = U_{0m}U_0', \; W = W_m W', \tag{6}$$

$$u_0 = U_{0m}u_0', \; v_0 = \frac{\delta}{L}U_{0m}v_0', \quad u_s = W_m u_s', \quad v_s = \frac{\delta_k}{L}W_m v_s' \quad (s = 1,2).$$

Here L is a certain length scale, and $\delta = \sqrt{\nu L/U_{0m}}$ is a quantity characterizing the thickness of the steady-flow region. The index m refers to the maximum value of a function, and the prime to the dimensionless form. The scales v_0 and v_s are determined by means of the equation of continuity.

A singular feature here is the introduction of a second scale with respect to the transverse coordinate, i.e., $\delta_k = \sqrt{2\nu/\omega}$. It has been verified in several theoretical and experimental studies [2-4] that δ_k is of the same order as the thickness of the region in which nonsteady motion in the boundary layer is concentrated. The introduction of δ_k (which is aptly called the thickness of the vibrational boundary layer) is required in order to bring the quantities v_s, $\partial u_s/\partial y$, $\partial^2 u_s/\partial y^2$ to dimensionless form.

In dimensionless variables (we drop the prime from now on), the system (5) takes the form

$$\left.\begin{aligned}
&1.\; u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} = U_0\frac{dU_0}{dx} + \frac{\partial^2 u_0}{\partial y^2} + \frac{1}{2}\beta^2 W\frac{dW}{dx} - \\
&\quad -\frac{1}{2}\beta^2 \operatorname{Re}\left(u_1\frac{\partial \bar{u}_1}{\partial x} + u_2\frac{\partial \bar{u}_2}{\partial x} + v_1\frac{\partial \bar{u}_1}{\partial \xi} + v_2\frac{\partial \bar{u}_2}{\partial \xi}\right), \\
&2.\; \frac{1}{2}\frac{\partial^2 u_1}{\partial \xi^2} - iu_1 = -iW + \alpha\frac{1}{\sqrt{2}}v_0\frac{\partial u_1}{\partial \xi} + \alpha^2\left[-W\frac{dU_0}{dx} - \right. \\
&\quad - U_0\frac{dW}{dx} + u_0\frac{\partial u_1}{dx} + u_1\frac{du_0}{dx} + \frac{1}{2}\beta\left(\bar{u}_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial \bar{u}_1}{\partial x} + \right. \\
&\quad \left.\left. + \bar{v}_1\frac{\partial u_2}{\partial \xi} + v_2\frac{\partial \bar{u}_1}{\partial \xi}\right)\right] + \alpha^3\sqrt{2}v_1\frac{\partial u_0}{\partial y}, \\
&3.\; \frac{1}{2}\frac{\partial^2 u_2}{\partial \xi^2} - 2iu_2 = \alpha\frac{1}{\sqrt{2}}v_0\frac{\partial u_2}{\partial \xi} + \alpha^2\left[u_0\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_0}{\partial x} + \right. \\
&\quad \left. + \frac{1}{2}\beta\left(-W\frac{dW}{dx} + u_1\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_1}{\partial \xi}\right)\right] + \alpha^3\sqrt{2}v_2\frac{\partial u_0}{\partial y}, \\
&4.\; u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_2}{\partial \xi} + v_2\frac{\partial u_1}{\partial \xi} = 0, \\
&5.\; u_2\frac{\partial u_2}{\partial x} + v_2\frac{\partial u_2}{\partial \xi} = 0, \\
&6.\; \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0, \\
&7{,}8.\; \frac{\partial u_s}{\partial x} + \frac{\partial v_s}{\partial \xi} = 0 \quad (s = 1,2).
\end{aligned}\right\} \tag{7}$$

The operators N_p are written out in expanded form in this system. Also, the following notation is introduced: $\alpha = \sqrt{U_{om}/\omega L} = 1/\sqrt{Sh}$ is a parameter of the problem, related to the Strouhal number, and $\beta = W_m/U_{om}$ is a parameter characterizing the ratio of the velocities of the vibrational and steady-flow motions.

The system (7) comprises the fundamental system of equations for the problem in question. It has the advantage over the system (1) of complete conditionality, insofar as the need for an initial condition is obviated (u_s and v_s do not depend on the time). However, it is impossible to obtain an exact solution of the system (7) in the general case, because the number of equations in it is greater than the number of unknown functions.

In the case $\alpha < 1$ ($Sh > 1$), the system (7), which contains terms with different powers of the parameter α, admits approximate solution. But if $\alpha > 1$ ($Sh < 1$), then quasisteady-state methods are clearly applicable.

Solution of the Fundamental System of Equations in the Case Sh > 1. We seek a solution of the system (7) in the form of series segments

$$u_s = u_s^{(0)} + u_s^{(1)}\alpha, \quad v_s = v_s^{(0)} + v_s^{(1)}\alpha, \quad s = 0,1,2, \tag{8}$$

i.e., we investigate motions with Strouhal numbers such that terms of order α^2, α^3, etc., can be neglected. We substitute the expansions (8) into the system (7) and equate expressions for identical powers of α. This process "decomposes" the system into equations, each of which can be solved in succession. We note that if the result of solution of an equation in the succession is $u_s^{(q)} \equiv 0$ ($q = 0, 1$), then $u_s^{(q)}$ is not included in the subsequent equations.

Equating terms containing α^0, and then those containing α^1, we obtain [the fourth and fifth equations in the system (7) are not used; we will show later that they are satisfied identically]

$$\left.\begin{aligned}
&1.\ \frac{1}{2}\frac{\partial^2 u_1^{(0)}}{\partial \xi^2} - iu_1^{(0)} = -iW,\\
&2.\ u_0^{(0)}\frac{\partial u_0^{(0)}}{\partial x} + v_0^{(0)}\frac{\partial u_0^{(0)}}{\partial y} = U_0\frac{dU_0}{dx} + \frac{\partial^2 u_0^{(0)}}{\partial y^2} +\\
&\quad + \frac{1}{2}\beta^2 W\frac{dW}{dx} - \frac{1}{2}\beta^2\,\mathrm{Re}\left(u_1^{(0)}\frac{\partial \bar{u}_1^{(0)}}{\partial x} + v_1^{(0)}\frac{\partial \bar{u}_1^{(0)}}{\partial \xi}\right),\\
&3.\ \frac{1}{2}\frac{\partial^2 u_1^{(1)}}{\partial \xi^2} - iu_1^{(1)} = \frac{1}{\sqrt{2}}v_0^{(0)}\frac{\partial u_1^{(0)}}{\partial \xi},\\
&4.\ u_0^{(0)}\frac{\partial u_0^{(1)}}{\partial x} + u_0^{(1)}\frac{\partial u_0^{(0)}}{\partial x} + v_0^{(0)}\frac{\partial u_0^{(1)}}{\partial y} + v_0^{(1)}\frac{\partial u_0^{(0)}}{\partial y} =\\
&\quad = \frac{\partial^2 u_0^{(1)}}{\partial y^2} - \frac{1}{2}\beta^2\,\mathrm{Re}\left(u_1^{(0)}\frac{\partial \bar{u}_1^{(1)}}{\partial x} + u_1^{(1)}\frac{\partial \bar{u}_1^{(0)}}{\partial x} + \right.\\
&\quad \left. + v_1^{(0)}\frac{\partial \bar{u}_1^{(1)}}{\partial \xi} + v_1^{(1)}\frac{\partial \bar{u}_1^{(0)}}{\partial \xi}\right).
\end{aligned}\right\} \tag{9}$$

We rewrite the boundary conditions in the form $u_s^{(q)} = 0$, $v_s^{(q)} = 0$ for $y = 0$; $u_0^{(o)} \to 1$, $u_1^{(o)} \to 1$ for $y \to \infty$; the remaining values are $u_s^{(q)} \to 0$ for $y \to \infty$; $u = f(y, t)$ for $x = x_f$. We bear in mind that the sixth through eighth equations of the system (7) are used to find the functions $v_s^{(q)}$.

The system (9) contains two types of equations. The first type includes the second and fourth in the system. They can be solved by the parametric method. The second equation has previously [5] been reduced to a universal form, i.e., in such a form that its solution does not require knowledge of the specific form of the functions $U_0(x)$ and $W(x)$. The first and third equations in the system (9) are of the second type. They can be treated as ordinary differential equations in the coordinate ξ with known right-hand sides. Hence, equations of the second type are universal in the stated sense. To find their particular solutions, it is useful to program the method of variation of arbitrary constants on a computer. The system (9) contains two more linear equations (which are omitted for brevity) admitting the trivial

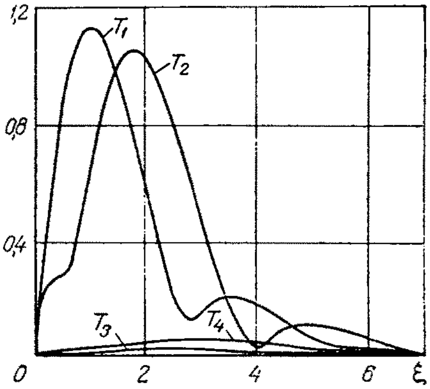

Fig. 1. Functions T_1, T_2, T_3, T_4 versus coordinate $\xi = y/\delta_k$ (all quantities are dimensionless).

solutions

$$u_2^{(0)} \equiv 0, \quad u_2^{(1)} \equiv 0. \tag{10}$$

We have thus shown that the parametric method is applicable for the solution of all equations in the system (9). Using the expansions (3) and (8), we can readily construct u and v from the values obtained for $u_s^{(q)}$ and $v_s^{(q)}$. This completes the solution of the stated problem. All that remains now is to note that the solution so obtained for the system (9) also satisfies the unused fourth and fifth equations of the system (7); see Eq. (10).

The foregoing scheme for the solution of the problem generalizes the conventional methods used for the analysis of a boundary layer. Thus, the first through the third equations of the system (9), which replace the system when $Sh \to \infty$, are identical to the computational scheme of the method of Lin [1]. If we put $dW/dx \equiv 0$ in the system (9) written with the inclusion of terms of order α^0, α^1, and α^2, we obtain the same computational scheme as in the method of Hill and Stenning [3].

Vibrations of a Cylinder in a Rest Fluid. As a simple example of the application of the proposed theory, we consider the vibrations of a cylinder in a fluid at rest. In this case, $U = W(x)\cos(\omega t)$. A singular aspect of this problem is the introduction of only one longitudinal velocity scale W_m and one transverse-coordinate scale δ_k. With the exception of these changes, the scheme of the solution in this case is analogous to that described above. We now give certain results of analytical calculations, at first including terms of order α^0, α^1, and α^2:

$$\begin{gathered}
u = \mathrm{Re}\,[u_1^{(0)}\exp(i\omega t)] + \frac{1}{\mathrm{Sh}_k}\ \mathrm{Re}\ [u_0^{(2)} + u_2^{(2)}\exp(2i\omega t)], \\
u_1^{(0)} = W\,[1 - \exp(-(1+i)\xi)], \qquad u_0^{(2)} = \frac{dW}{dx}\,\mathrm{Re}\Big[-\frac{3}{4} \\
+ \frac{1}{4}\exp(-2\xi) + \left(2\sin\xi + \frac{1}{2}\cos\xi\right)\exp(-\xi) + \frac{\xi}{2}(\sin\xi - \cos\xi)\exp(-\xi)\Big], \\
u_2^{(2)} = \frac{dW}{dx}\left[-\frac{i}{2}\exp(-\sqrt{2}(1+i)\xi) + \left(\frac{1-i}{2}\xi + \frac{i}{2}\right)\exp(-(1+i)\xi)\right], \\
\mathrm{Sh}_k = \frac{\omega L}{W_m}.
\end{gathered} \tag{11}$$

The remaining quantities $u_s^{(q)} \equiv 0$. The final solution and initial equations exactly correspond to the method of Rayleigh [2] (see also [6]). Our proposed method makes it possible to solve the problem more precisely and to include, for example, additional terms of order α^3 and α^4. The refinement (additional term) of the longitudinal velocity has the form

$$u_{\text{add}} = \frac{1}{\mathrm{Sh}_k^2}\ \mathrm{Re}\ [u_1^{(4)}\exp(i\omega t) + u_3^{(4)}\exp(3i\omega t)],$$

$$u_1^{(4)} = \frac{d}{dx}\left(W\frac{dW}{dx}\right)T_1(\xi) + \left(\frac{dW}{dx}\right)^2 T_2(\xi), \tag{12}$$

$$u_3^{(4)} = \frac{d}{dx}\left(W\frac{dW}{dx}\right)T_3(\xi) + \left(\frac{dW}{dx}\right)^2 T_4(\xi).$$

Equations (11) and (12) are written in dimensionless form. The dependences of the functions T_1, T_2, T_3, T_4 on the coordinate ξ are determined in the analytical solution of equations of the second type in a system analogous to (9) and are given in Fig. 1. The functions T_1 and T_2 obtained by consideration of the interaction of the steady flow with vibrations of frequency ω are of decisive significance in the refinement. The second and higher harmonics of the longitudinal velocity are small in comparison with the first, and so the assumption of rapid convergence of the Fourier series (3) is justified. This conclusion can be extended to the general case in which $U(x, t) = U_0(x) + W(x)\cos(\omega t)$, since the order of the second and higher harmonics of u is determined solely by the function W(x); see the system (7).

We now derive an expression for estimating the error of determination of the velocity u by the set of equations (11) and (12). Inasmuch as the values of the second and higher harmonics are at least an order of magnitude smaller than the values of the zeroth and first harmonics, the error is determined by the functions $u_0^{(q)}$ and $u_1^{(q)}$ ($q = 0, 1, 2, \ldots$). We set the largest values of these functions equal to unity and sum the remainders of the series (8). Then

$$\Delta|u| \leqslant \frac{1}{\mathrm{Sh}_k^2(\mathrm{Sh}_k - 1)}. \tag{13}$$

For example, given $\mathrm{Sh}_k = 3$, the error $\Delta|u| \leq 5.6\%$, and the refinement relative to the Rayleigh method [expression (11)] is equal to 11.1%.

The inclusion of more terms in the expansions (8) can be realized without fundamental difficulties and increases the precision of the proposed method.

NOTATION

x, y, longitudinal and transverse coordinates in the boundary layer; u, v, projections of the velocity in the boundary layer onto the x and y axes, respectively; t, time; ω, frequency of velocity oscillations at the outer boundary of the boundary layer; ν, kinematic viscosity coefficient; f(y, t), initial velocity profile in the boundary layer; Re, Im, real and imaginary parts of a complex number.

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